Quantum regression theorem for non-Markovian Lindblad equations

Adrián A. Budini[1,2]
[1] *Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, 01187 Dresden, Germany*
[2] *Instituto de Biocomputación y Física de Sistemas Complejos, Universidad de Zaragoza, Corona de Aragón 42, (50009) Zaragoza, Spain**

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We find the conditions under which a quantum regression theorem can be assumed valid for non-Markovian master equations consisting in Lindblad superoperators with memory kernels. Our considerations are based on a generalized Born-Markov approximation, which allows us to obtain our results from an underlying Hamiltonian description. We demonstrate that a non-Markovian quantum regression theorem can only be granted in a stationary regime if the dynamics satisfies a quantum detailed balance condition. As an example we study the correlations of a two level system embedded in a complex structured reservoir and driven by an external coherent field.

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I. INTRODUCTION

In many areas of physics one is confronted with the description of small quantum systems interacting with an uncontrollable environment. This situation is well understood when the reduced system dynamics follows a (completely positive) Markovian evolution [1, 2, 3, 4, 5, 6].

One of the cornerstone of the theory of Markovian open quantum systems is the quantum regression theorem (QRT). This theorem, originally proposed by Lax [7], allows to calculate multiple-time operators correlation functions from the knowledge of single-time expectation values, which in turn implies the knowledge of the density matrix evolution [4, 5, 6]. The importance of this theorem comes from the physical information contained in the operator correlations. In fact, in a stationary regime, it is possible to relate the Fourier transform of these objects with the spectrum of the decay process [3]. Furthermore, in radiant systems, the statistic of the scattered field can be described through system operator correlations [4, 5, 6].

Another central cornerstone of non-equilibrium quantum Markovian dynamics is the quantum detailed balance condition, which imposes severe symmetry properties on the operator correlations structure. While in classical stochastic processes this condition has a clear meaning in terms of transitions between the available states of the system [8, 9], in quantum dissipative systems this condition relies in the time reversal property of the underlying stationary microscopic Hamiltonian evolution [10, 11, 12, 13, 14, 15]. The breakdown of this condition has direct experimental implications [16].

Although the applicability of the Markovian approximation range over many physical situations [1, 2, 3, 4, 5, 6], there exist several real systems whose dynamics present strong departures from it. Remarkable examples are anomalous intermittent fluorescence in quantum dots [17, 18, 19, 20], the presence of $1/f$ noise in phase and charge superconducting qubits [21, 22], and band gap materials [23, 24].

Consistently with the existence of experimental situations that can not be described by a Markovian evolution, in the context of different approaches recent effort was dedicated to characterize non-Markovian operator correlation dynamics [25, 26, 27].

While the description of non-Markovian processes may depends on each specific situation, there exists an increasing interest in describing these kind of processes by introducing memory contributions in standard Lindblad evolutions [28, 29, 30, 31, 32, 33, 34, 35, 36]. This procedure provide easy manageable equations. Nevertheless, this technique does not have associated a rule for calculating operator correlations.

In this paper we explore the possibility of establishing a QRT for non-Markovian master equations that can be cast in the form of Lindblad equations with memory contributions [28, 29, 30, 31, 32, 33, 34, 35, 36]. We will base our considerations in a generalized Born-Markov approximation (GBMA) [36], which allows us to develop our results from an underlying microscopic Hamiltonian description.

The paper is outlined as follows. In Section II we review the derivation of the GBMA from a full microscopic description. Based on this approach, in Section III we search the conditions under which a non-Markovian QRT can be established. In Section IV we relate the non-Markovian QRT with a detailed balance condition. In Section V we exemplify our theoretical results by analyzing the correlation dynamics of a two level system embedded in a complex thermal environment described in a GBMA. In section VI we give the conclusions.

II. GENERALIZED BORN-MARKOV APPROXIMATION

The GBMA applies for complex structured environments whose action over the system can be well approx-

*present address

imated by a direct sum of sub-reservoirs, each one being able to induce by itself a Markovian system dynamics. Under this condition, the system evolution can be written as a Lindblad equation characterized by a random dissipative rate [36].

Here, we review the microscopic derivation of this approximation by using a well known projector operator technique [37, 38]. This equivalent derivation is useful for clarifying that the GBMA is not restricted to a second order approximation. In fact, the projector technique provides a rigorous procedure that allows to obtain the system dynamics up to any desired order in the interaction Hamiltonian.

We assume a full microscopic Hamiltonian description of the interaction of a system S with its environment B

$$H_T = H_S + H_B + H_I. \tag{1}$$

Here, H_S and H_B correspond to the system and bath Hamiltonians respectively. The term $H_I = q_S \otimes Q_B$ describes their mutual interaction, with the operators q_S and Q_B acting on the system and bath Hilbert spaces respectively.

The system density matrix follows after eliminating the environment degrees of freedom, $\rho_S(t) = \mathrm{Tr}_B\{\rho_T(t)\}$, where the total density matrix $\rho_T(t)$ evolves as

$$\frac{d\rho_T(t)}{dt} = \frac{-i}{\hbar}[H_T, \rho_T(t)] \equiv \mathcal{L}_T[\rho_T(t)]. \tag{2}$$

The GBMA [36] can be derived by introducing the projector $\mathcal{P}$ defined by

$$\mathcal{P}\rho_T(t) \equiv \sum_R \rho_R(t) \otimes \Xi_R, \tag{3}$$

where Ξ_R is given by

$$\Xi_R \equiv \Pi_R \rho_B \Pi_R, \tag{4}$$

with ρ_B being the stationary state of the bath, while the system states $\rho_R(t)$ are defined by

$$\rho_R(t) \equiv \frac{\mathrm{Tr}_B\{\Pi_R \rho_T(t) \Pi_R\}}{\mathrm{Tr}_B\{\Pi_R \rho_B \Pi_R\}}. \tag{5}$$

Here, we have introduced a set of projectors $\Pi_R = \sum_{\{\epsilon_R\}} |\epsilon_R\rangle \langle\epsilon_R|$, which provides an orthogonal decomposition of the unit operator $[I_B]$ in the Hilbert space of the bath, $\sum_R \Pi_R = I_B$, with $\Pi_R \Pi_{R'} = \Pi_R \delta_{R,R'}$. The full set of states $|\epsilon_R\rangle$ corresponds to the base where ρ_B is diagonal, which implies $\sum_R \Xi_R = \rho_B$.

It is easy to realize that $\mathcal{P}^2 = \mathcal{P}$. In physical terms, this projector takes in account that each bath-subspace associated to the projectors Π_R induces a different system dynamics, each one represented by the states $\rho_R(t)$. On the other hand, notice that the standard projector $\mathcal{P}\rho_T(t) = \mathrm{Tr}_B\{\rho_T(t)\} \otimes \rho_B = \rho_S(t) \otimes \rho_B$ [37, 38], is recuperated when all the states $\rho_R(t)$ have the same dynamics.

From Eq. (3), the system density matrix follows as

$$\rho_S(t) = \mathrm{Tr}_B\{\mathcal{P}\rho_T(t)\} = \sum_R P_R \rho_R(t) \equiv \langle \rho_R(t) \rangle. \tag{6}$$

This equation defines the system state as an average over the density matrixes $\rho_R(t)$, each one participating with weight P_R. These parameters are defined by the weight of each subspace in the full stationary bath state

$$P_R = \mathrm{Tr}_B\{\Xi_R\} = \mathrm{Tr}_B\{\Pi_R \rho_B\} = \sum_{\{\epsilon_R\}} \langle \epsilon_R | \rho_B | \epsilon_R \rangle, \tag{7}$$

which in consequence satisfy $\sum_R P_R = 1$.

By writing the evolution Eq. (2) in an interaction representation, and splitting the full dynamics in contributions $\mathcal{P}\rho_T(t)$ and $\mathcal{Q}\rho_T(t)$, where $\mathcal{Q} = 1 - \mathcal{P}$, up to second order in the interaction Hamiltonian it follows [37, 38]

$$\frac{d\mathcal{P}\rho_T(t)}{dt} = \int_0^t dt' \mathcal{P}\mathcal{L}_T(t)\mathcal{L}_T(t')\mathcal{P}\rho_T(t'), \tag{8}$$

where $\mathcal{L}_T(t)$ is the total Liouville operator in a interaction representation. Here, we have assumed an uncorrelated initial state, $\rho_T(0) = \rho_S(0) \otimes \rho_B$.

By assuming an interaction Hamiltonian with a direct sum structure

$$H_I = H_{I_1} \oplus H_{I_2} \cdots \oplus H_{I_R} \oplus H_{I_{R+1}} \cdots, \tag{9}$$

where each term satisfies $H_{I_R} = \Pi_R H_I \Pi_R$, from Eq. (8) it follows that each state $\rho_R(t)$, in a Schrödinger representation, evolves as

$$\frac{d\rho_R(t)}{dt} = \frac{-i}{\hbar}[H_S, \rho_R(t)] - \left(\frac{1}{\hbar}\right)^2 \int_0^\infty dt' \, \mathrm{Tr}_{B_R}\{[H_{I_R}, [H_{I_R}(-t'), \rho_R(t) \otimes \rho_{B_R}]]. \tag{10}$$

with $\rho_{B_R} \equiv \Xi_R / P_R$, and where $\mathrm{Tr}_{B_R}\{\bullet\} \equiv \mathrm{Tr}_B\{\Pi_R \bullet \Pi_R\}$. The corresponding initial condition reads $\rho_R(0) = \rho_S(0)$, which follows from Eq. (5). Furthermore, in this evolution we have introduced a Markov approximation, which applies when each bath subspace corresponding to the projectors Π_R defines a Markovian sub-environment.

The evolution Eq. (10), disregarding transients of the order of the sub-bath correlation time, can be always well approximated by a Lindblad equation [1]

$$\frac{d\rho_R(t)}{dt} = \mathcal{L}_H[\rho_R(t)] + \gamma_R \mathcal{L}[\rho_R(t)], \tag{11}$$

where $\mathcal{L}_H[\bullet] = -(i/\hbar)[H_S, \bullet]$, and the dissipative contribution is defined by a Lindblad superoperator [1]

$$\mathcal{L}[\bullet] = \frac{1}{2} \sum_{\alpha\beta} a_{\alpha\beta}([V_\alpha, \bullet V_\beta^\dagger] + [V_\alpha \bullet, V_\beta^\dagger]). \tag{12}$$

Here, the set of system operators $\{V_\alpha\}$ and the dimensionless Hermitian matrix $a_{\alpha\beta}$ depend on the underlying microscopic interaction. The rates γ_R follow from a

Fermi golden rule when applied to the manifold of states $\{|\epsilon_R\rangle\}$ that define each Markovian sub-reservoir.

While the density matrixes $\rho_R(t)$ follow a Markovian evolution, the system state $\rho_S(t)$ evolves with a completely positive [1, 2] non-Markovian evolution, property inherited from the random Lindblad structure Eq. (11). The average of this equation over the set $\{\gamma_R, P_R\}$ can be performed in a Laplace domain, from where it follows

$$\frac{d\rho_S(t)}{dt} = \mathcal{L}_H[\rho_S(t)] + \int_0^t d\tau\, \mathbb{L}(t-\tau)[\rho_S(\tau)], \qquad (13)$$

where the superoperator $\mathbb{L}(t)$ is defined by the relation

$$\langle G_R(u)\gamma_R\mathcal{L}\rangle[\bullet] = \langle G_R(u)\rangle\mathbb{L}(u)[\bullet]. \qquad (14)$$

Here, u is the Laplace variable and $G_R(u)$ is the Markovian propagator of each state $\rho_R(t)$, i.e., $G_R(u) \equiv [u - (\mathcal{L}_H + \gamma_R\mathcal{L})]^{-1}$. Depending on the set $\{\gamma_R, P_R\}$, which specify the complex environment, Eq. (13) may lead to a reach variety of system decay behaviors as well as to many different structures of *non-local Lindblad equations.* In fact, in general $\mathbb{L}(u)$ consists in a sum of Lindblad terms, each one characterized a different memory kernel.

The structure of the superoperator $\mathbb{L}(u)$ can be simplified in an effective approximation [36], which consists of discarding the dependence introduced by the Lindblad superoperator $\mathcal{L}$ in the propagator $G_R(u)$, i.e., $\mathcal{L}_R \to -\mathrm{I}$. From Eq. (14) it follows the approximated solution $\mathbb{L}(u) \simeq K(u - \mathcal{L}_H)\mathcal{L}$, which implies the evolution

$$\frac{d\rho_S(t)}{dt} \simeq \mathcal{L}_H[\rho_S(t)] + \int_0^t d\tau K(t-\tau)e^{(t-\tau)\mathcal{L}_H}\mathcal{L}[\rho_S(\tau)]. \qquad (15)$$

with $K(u) = \langle\gamma_R(u+\gamma_R)^{-1}\rangle\langle(u+\gamma_R)^{-1}\rangle^{-1}$. If the time scale of the unitary dynamics is larger than the time scale of the memory kernel, the unitary contribution can be discarded leading to a single memory Lindblad equation. We remark that structures similar to Eq. (15) were obtained in the context of other approaches [29, 30, 31, 32, 33, 34, 35]. The GBMA, here defined through the projector Eq. (3), allows us to associate an underlying well defined microscopic description to these kind of equations.

III. QUANTUM REGRESSION THEOREM

For Markovian master equations the QRT [4, 5, 6] provides a direct relation between the evolution of the expectation values of system observables and their corresponding correlation functions. Here we will explore the possibility of formulating an equivalent relation when the system dynamics can be described through the GBMA.

A. Random rate formulation for operators correlations

Let us introduce a complete set of operators $\{A_\mu\}$ of the system, collected into a vector $\mathbf{A}$, and consider the expectation values

$$\overline{\mathbf{A}(t)} \equiv \mathrm{Tr}_{SB}[\mathbf{A}(t)\rho_T(0)], \qquad (16)$$

as well as the correlation functions

$$\overline{O(t)\mathbf{A}(t+\tau)} \equiv \mathrm{Tr}_{SB}[O(t)\mathbf{A}(t+\tau)\rho_T(0)], \qquad (17)$$

where $O(t)$ is an arbitrary system operator. The time dependence of the operators refers to a Heisenberg representation with respect to the total Hamiltonian Eq. (1), i.e., $O(t) = \exp[(i/\hbar)tH_T]O(0)\exp[-(i/\hbar)tH_T]$.

From Eq. (6), we can write the expectation values as an average over the solutions corresponding to each rate

$$\overline{\mathbf{A}(t)} = \langle \mathrm{Tr}_S[\mathbf{A}(0)\rho_R(t)]\rangle \equiv \langle\overline{\mathbf{A}(t)}_R\rangle. \qquad (18)$$

In order to work out the operator correlations, we first express the total initial density matrix as $\rho_T(0) = \exp[(i/\hbar)tH_T]\rho_T(t)\exp[-(i/\hbar)tH_T]$. Then, by using the cyclic property of the trace, from Eq. (17) we obtain

$$\overline{O(t)\mathbf{A}(t+\tau)} = \mathrm{Tr}_S\{\mathbf{A}(0)\mathrm{Tr}_B[O_{SB}(\tau)]\}, \qquad (19)$$

where the operator $O_{SB}(\tau)$ satisfies

$$\frac{d}{d\tau}O_{SB}(\tau) = -\frac{i}{\hbar}[H_T, O_{SB}(\tau)], \qquad (20)$$

with $O_{SB}(\tau)|_{\tau=0} = \rho_T(t)O(0)$. This system-bath operator evolves as the total density matrix, Eq. (2). On the other hand, Eq. (3) allows us to write the initial condition as $O_{SB}(\tau)|_{\tau=0} \approx \sum_R[\rho_R(t)O(0)] \otimes \Xi_R$ [39]. Therefore, the reduced dynamics of $O_{SB}(\tau)$ can also be described in a GBMA, which deliver

$$\mathrm{Tr}_B[O_{SB}(\tau)] = \langle\exp[(\mathcal{L}_H + \mathcal{L}_R)\tau]\rho_R(t)\rangle O(0), \qquad (21)$$

where, for shortening the notation we defined $\mathcal{L}_R \equiv \gamma_R\mathcal{L}$. From Eq. (19), it follows

$$\begin{aligned}\overline{O(t)\mathbf{A}(t+\tau)} &= \langle\mathrm{Tr}_S\{\mathbf{A}(0)e^{(\mathcal{L}_H+\mathcal{L}_R)\tau}[\rho_R(t)O(0)]\}\rangle \\ &\equiv \langle\overline{O(t)\mathbf{A}(t+\tau)}_R\rangle. \end{aligned} \qquad (22)$$

This expression is an average over the random set $\{\gamma_R, P_R\}$ of the corresponding Markovian correlation expressions [4]. This characteristic provides us a central result, which allows us to extend the averaging procedure [Eq. (6)] corresponding to the GBMA for operator correlations as well. In fact, *higher correlations operators can also be obtained as an average, over the random rate set, of the Markovian expressions corresponding to each state* $\rho_R(t)$. For example, using the same steps as before, for arbitrary system operators O_1 and O_2, it is possible to obtain

$$\begin{aligned}\overline{O_1(t)\mathbf{A}(t+\tau)O_2(t)} &= \langle\mathrm{Tr}_S\{\mathbf{A}e^{(\mathcal{L}_H+\mathcal{L}_R)\tau}[O_2\rho_R(t)O_1]\}\rangle \\ &\equiv \langle\overline{O_1(t)\mathbf{A}(t+\tau)O_2(t)}_R\rangle, \end{aligned} \qquad (23)$$

which also correspond to an average over the corresponding Markovian dynamics [4].

B. Expectation and correlation evolution

From the previous result, we can write the evolution of both, expectation values and correlations, as an average over the random rate set

$$\frac{d}{dt}\overline{\mathbf{A}(t)} = \langle \hat{\mathbf{M}}_R \overline{\mathbf{A}(t)}_R \rangle, \tag{24a}$$

$$\frac{d}{d\tau}\overline{O(t)\mathbf{A}(t+\tau)} = \langle \hat{\mathbf{M}}_R \overline{O(t)\mathbf{A}(t+\tau)}_R \rangle. \tag{24b}$$

Here, the matrix $\hat{\mathbf{M}}_R$ acts on the indices of $\mathbf{A}$ and is defined by the condition

$$\mathrm{Tr}_S\{\mathbf{A}(\mathcal{L}_H + \mathcal{L}_R)[O]\} = \hat{\mathbf{M}}_R \mathrm{Tr}_S\{\mathbf{A}O\}. \tag{25}$$

When γ_R is fixed, the evolution equations (24a) for expectation values and (24b) for correlation functions are identical, which recovers the *QRT for Markovian dynamics*. In the non-Markovian case, however, both equations still involve the average over the dissipation rate.

As for the density matrix [36], the averaged evolutions can be worked out in the Laplace domain. The expectation value can be expressed as $\overline{\mathbf{A}(u)} = \langle \hat{\mathbf{G}}_R(u)\overline{\mathbf{A}(0)}\rangle$, with the matrix propagator $\hat{\mathbf{G}}_R(u) \equiv (u + \hat{\mathbf{M}}_R)^{-1}$. After introducing the identity operator in the form $\overline{\mathbf{A}(u)} = \langle \hat{\mathbf{G}}_R(u)(u+\hat{\mathbf{M}}_R)\rangle^{-1}\langle \hat{\mathbf{G}}_R(u)\overline{\mathbf{A}(0)}\rangle$, we arrive to the deterministic closed evolution

$$\frac{d}{dt}\overline{\mathbf{A}(t)} = -\int_0^t dt' \hat{\mathbb{M}}(t-t')\overline{\mathbf{A}(t')}. \tag{26a}$$

Using a similar procedure, for the correlation we get

$$\frac{d}{d\tau}\overline{O(t)\mathbf{A}(t+\tau)} = -\int_0^\tau dt'\ \hat{\mathbb{M}}(\tau-t')\overline{O(t)\mathbf{A}(t+t')} + \mathbf{I}(t,\tau). \tag{26b}$$

The deterministic kernel matrix $\hat{\mathbb{M}}(t)$ fulfills the equation

$$\hat{\mathbb{M}}(u) = \langle \hat{\mathbf{G}}_R(u)\rangle^{-1}\langle \hat{\mathbf{G}}_R(u)\hat{\mathbf{M}}_R\rangle, \tag{27}$$

while the inhomogeneous term $\mathbf{I}(t,\tau)$ is defined by

$$\mathbf{I}(t,u) = \langle \hat{\mathbf{G}}_R(u)\rangle^{-1}\langle \hat{\mathbf{G}}_R(u)\overline{O(t)\mathbf{A}(t)}_R\rangle - \langle \overline{O(t)\mathbf{A}(t)}_R\rangle. \tag{28}$$

Besides that Eq. (24b) has the same structure as Eq. (24a), the inhomogeneous term is only present in the correlation evolution, Eq. (26b). $\mathbf{I}(t,\tau)$ arise because the initial condition of each contribution in Eq. (24b) is correlated with respect to its propagator. In fact, notice that both $\hat{\mathbf{G}}_R(u)$ and $\overline{O(t)\mathbf{A}(t)}_R$ depend on γ_R, which implies that theses objects are correlated with respect to the random rate statistics. The dependence of $\overline{O(t)\mathbf{A}(t)}_R$ on γ_R follows from $\overline{O(t)\mathbf{A}(t)}_R = \mathrm{Tr}_S\{O(0)\mathbf{A}(0)\rho_R(t)]$. On the other hand, as Eq. (24a) is defined with initial conditions fixed at $t=0$, its initial condition $\overline{\mathbf{A}(0)}_R$ does not depends on γ_R, which in turn implies that the inhomogeneous term is not present in the averaged evolution Eq. (26a).

Due to the inhomogeneous term $\mathbf{I}(t,\tau)$, the QRT is not fulfilled in general. A non-Markovian QRT is only valid when this term vanish, which leads to the condition

$$\langle \hat{\mathbf{G}}_R(u)\overline{O(t)\mathbf{A}(t)}_R\rangle \overset{QRT}{=} \langle \hat{\mathbf{G}}_R(u)\rangle\langle \overline{O(t)\mathbf{A}(t)}_R\rangle \tag{29}$$

This equality is always satisfied for Markovian dynamics because the average over the dissipation rate is absent. We also note that *a non-Markovian QRT can be asymptotically valid if the stationary state $\rho_R^\infty \equiv \rho_R(\infty)$ does not depend on γ_R* [40]. In fact, in this situation $\lim_{t\to\infty}\overline{O(t)\mathbf{A}(t)}_R = \mathrm{Tr}_S\{O(0)\mathbf{A}(0)\rho_R^\infty]$ is independent of γ_R, and then the condition Eq. (29) is automatically satisfied. However, if the asymptotic state ρ_R^∞ depends on γ_R the inhomogeneous term will contribute at all times, even in the asymptotic regime, and the QRT is invalidated. The same condition is valid for higher operators correlations.

C. Non-Markovian dynamics

The evolution Eq. (26a) and (26b) can be formally integrated in the Laplace domain. For the expectation values we get

$$\overline{\mathbf{A}(t)} = \hat{\mathbb{G}}(t)\overline{\mathbf{A}(0)}, \tag{30a}$$

while for the correlations it follows

$$\overline{O(t)\mathbf{A}(t+\tau)} = \hat{\mathbb{G}}(\tau)\overline{O(t)\mathbf{A}(t)} + \mathbf{F}(t,\tau). \tag{30b}$$

The non-Markovian propagator is defined by

$$\hat{\mathbb{G}}(u) = \frac{1}{u + \hat{\mathbb{M}}(u)}, \tag{31}$$

and the extra inhomogeneous term is

$$\mathbf{F}(t,\tau) = \langle \hat{\mathbf{G}}_R(\tau)\overline{O(t)\mathbf{A}(t)}_R\rangle - \langle \hat{\mathbf{G}}_R(\tau)\rangle\langle \overline{O(t)\mathbf{A}(t)}_R\rangle. \tag{32}$$

These expressions explicitly show that the departure from condition Eq. (29) measures the size of the dynamical effects which can not be captured by assuming valid the QRT. In fact, the QRT is fulfilled only when $\mathbf{F}(t,\tau)$ vanishes.

Eq. (30a) and (30b) are consistent with the averaging procedure over Markovian solutions. In fact, they can be expressed as $\overline{\mathbf{A}(t)} = \langle \hat{\mathbf{G}}_R(\tau)\rangle\overline{\mathbf{A}(0)}$, and for the correlations as

$$\overline{O(t)\mathbf{A}(t+\tau)} = \langle \hat{\mathbf{G}}_R(\tau)\overline{O(t)\mathbf{A}(t)}_R\rangle, \tag{33}$$

which in fact are an average over Markovian solutions.

D. Fluctuation operators

Of special interest is to study the correlation dynamics of fluctuation operators, which are defined as the departure from expectation values

$$\delta\mathbf{A}(t) \equiv \mathbf{A}(t) - \overline{\mathbf{A}(t)}, \tag{34a}$$

$$\delta O(t) \equiv O(t) - \overline{O(t)}. \tag{34b}$$

For Markovian dynamics the correlation of these operators also satisfies a QRT. These objects are relevant to split the spectrum, defined as the Fourier transform of the stationary correlations, in a coherent and incoherent components [4].

From Eq. (33) we can write

$$\begin{aligned}\overline{O(t)\mathbf{A}(t+\tau)} &= \langle \hat{\mathbf{G}}_R(\tau)\overline{\delta O(t)\delta\mathbf{A}(t+\tau)}_R\rangle \\ &\quad + \langle \overline{O(t)}_R\overline{\mathbf{A}(t+\tau)}_R\rangle. \end{aligned}\tag{35}$$

For Markovian evolutions, in the asymptotic time regime ($t\to\infty$), the first contribution can be associated with the incoherent spectrum component while the second one, after taking the extra limit $\tau\to\infty$, with the coherent spectrum part. After averaging over the random rate, these associations remains valid for the non-Markovian case. In particular, we note that the coherent component

$$\lim_{\substack{t\to\infty\\ \tau\to\infty}} \overline{O(t)\mathbf{A}(t+\tau)} = \langle \overline{O(\infty)}_R\ \overline{\mathbf{A}(\infty)}_R\rangle \tag{36}$$

is an average of the corresponding Markovian contributions.

The correlation Eq. (33) can also be written as

$$\overline{O(t)\mathbf{A}(t+\tau)} = \overline{\delta O(t)\delta\mathbf{A}(t+\tau)} + \overline{O(t)}\ \ \overline{\mathbf{A}(t+\tau)}. \tag{37}$$

As for the Markovian case, this expression follows immediately from the microscopic definition Eq. (17). From this relation and Eq. (35), we get

$$\overline{\delta O(t)\delta\mathbf{A}(t+\tau)} = \hat{\mathbb{G}}(\tau)\overline{\delta O(t)\delta\mathbf{A}(t)} + \delta\mathbf{F}(t,\tau). \tag{38}$$

Here, the first contribution follows from the QRT when assumed valid for fluctuations operators, and the second one measures the departure from it, being defined by

$$\begin{aligned}\delta\mathbf{F}(t,\tau) &= \langle \hat{\mathbf{G}}_R(\tau)\overline{\delta O(t)\delta\mathbf{A}(t)}_R\rangle - \langle \hat{\mathbf{G}}_R(\tau)\rangle\langle\overline{\delta O(t)\delta\mathbf{A}(t)}_R\rangle \\ &\quad + \langle \overline{O(t)}_R\overline{\mathbf{A}(t+\tau)}_R\rangle - \langle \overline{O(t)}_R\rangle\langle\overline{\mathbf{A}(t+\tau)}_R\rangle.\end{aligned}$$

As for operators, in the asymptotic regime the QRT is also valid for fluctuation operators if the stationary sate ρ_R^∞ does not depend on the random rate, which is fact implies $\delta\mathbf{F}(\infty,\tau)=0$.

From Eq. (37) and Eq. (38), we notice that by assuming valid the QRT, the coherent spectrum component reads

$$\lim_{\substack{t\to\infty\\ \tau\to\infty}} \overline{O(t)\mathbf{A}(t+\tau)} \overset{QRT}{=} \langle \overline{O(\infty)}_R\rangle\langle\overline{\mathbf{A}(\infty)}_R\rangle. \tag{39}$$

This expression and Eq. (36) indicate that the predictions of the QRT will differ from the exact dynamics not only in the transient dynamical behaviors but in general also in the asymptotic correlation values. In the next sections, we will use the difference between these two expressions as a measure of the deviation from the validity of the QRT in the stationary regime.

IV. DETAILED BALANCE CONDITION

In the context of the GBMA, in the previous section we have demonstrated that the QRT can be assumed valid in an asymptotic regime if the stationary state ρ_R^∞ corresponding to each Markovian contribution does not depends on the random rate. Here, we will find an equivalent condition which does not depends on the approximations used to arrive to the non-Markovian Lindblad equation. We will demonstrate that the previous result can be associated with a quantum detailed balance condition [10, 11, 12, 13, 14, 15], which in turn is related with the microrreversibility of the underlying microscopic dynamics [10].

A. Classical conditions

The concept of detailed balance is well established for classical population master equations [8]

$$\frac{dp_n(t)}{dt} = \gamma_{cl}\Big\{\sum_m g_{nm}p_m(t) - \sum_m g_{mn}p_n(t)\Big\}, \tag{40}$$

where $\gamma_{cl}g_{nm}$ define the hopping rates. The classical detailed balance condition reads

$$\gamma_{cl}g_{nm}p_m(\infty) = \gamma_{cl}g_{mn}p_n(\infty), \tag{41}$$

which has an immediate interpretation in terms of the available stationary transitions. We note that these relations does not depend on the global rate γ_{cl}. Thus, they impose strong relations between the dimensionless hopping coefficients $\{g_{nm}\}$ and the stationary populations $\{p_n(\infty)\}$. In particular, it is possible to prove that when the stationary state depends on an arbitrary continuous parameter ε, $\{p_m(\infty,\varepsilon)\}$, the hopping coefficients must also to depend on that parameter, $\{g_{mn}(\varepsilon)\}$. If this is not the case, the detailed balance condition is violated [41]. This result can be extended to quantum master equations, ε being the random rate, establishing a strong relation between the validity of the QRT for non-Markovian dynamics and the detailed balance condition.

B. Quantum Markovian conditions

The detailed balance condition can be generalized for quantum dynamics from the time reversal property [10]

of the stationary system-bath dynamics. For an open *Markovian* system, it can be written as an statement of time symmetry for *stationary* two-time operator correlations [10, 11, 12, 13]

$$\lim_{t\to\infty} \overline{O(t+\tau)\mathbf{A}(t)}_R = \lim_{t\to\infty} \overline{\widetilde{\mathbf{A}}(t+\tau)\widetilde{O}(t)}_R, \qquad (42)$$

where $\widetilde{O}(t)$ and $\widetilde{\mathbf{A}}(t)$ represent time-reversed operators [42]. From this equation [13], it is possible to write an equivalent formulation in the Laplace domain as [43]

$$\rho_R^\infty \frac{1}{u-(\mathcal{L}_H^\# + \mathcal{L}_R^\#)}[\bullet] = \frac{1}{u-(\widetilde{\mathcal{L}}_H + \widetilde{\mathcal{L}}_R)}[\widetilde{\rho}_R^\infty \bullet]. \qquad (43)$$

We have introduced dual and time reversed superoperators [10, 11, 12, 13], which respectively are defined by $\mathrm{Tr}_S\{O\mathcal{L}[\rho]\} = \mathrm{Tr}_S\{\rho\mathcal{L}^\#[O]\}$, and by $\widetilde{\mathcal{L}[O]} = \widetilde{\mathcal{L}}[\widetilde{O}]$ [44]. Equation (43) is equivalent to the conditions

$$\widetilde{\rho}_R^\infty = \rho_R^\infty, \qquad (44a)$$
$$H_S\rho_R^\infty = \rho_R^\infty H_S, \qquad (44b)$$
$$\gamma_R\rho_R^\infty\mathcal{L}^\#[\bullet] = \gamma_R\widetilde{\mathcal{L}}[\rho_R^\infty\bullet]. \qquad (44c)$$

From the second equation and the stationary condition, $\{\mathcal{L}_H + \gamma_R\mathcal{L}\}[\rho_R^\infty] = 0$, it follows $\mathcal{L}[\rho_R^\infty] = 0$. This condition cannot be satisfied consistently if the stationary state ρ_R^∞ depends on γ_R. In fact, the superoperator $\mathcal{L}$ does not has a continuous parametrized degenerate null eigen-operator. An equivalent conclusion can be obtained from the third relation. Then, we deduce that whenever ρ_R^∞ depends on the random rate γ_R the detailed balance condition is violated. Therefore, we can affirm that *if the underlying Markovian evolution of $\rho_R(t)$ satisfies the quantum detailed balance conditions, Eq. (44), the non-Markovian QRT is valid in the asymptotic regime.* Equivalently, this statement indicates that when the non-Markovian QRT is not fulfilled, the detailed balance conditions Eq. (44) are also not satisfied.

C. Quantum non-Markovian conditions

The microrreversibility condition Eq. (42) can be trivially extended to the non-Markovian dynamics as

$$\lim_{t\to\infty} \overline{O(t+\tau)\mathbf{A}(t)} = \lim_{t\to\infty} \overline{\widetilde{\mathbf{A}}(t+\tau)\widetilde{O}(t)}. \qquad (45)$$

After applying the averaging procedure, from Eqs. (42) and (43), we get the equivalent condition

$$\left\langle \rho_R^\infty \frac{1}{u-(\mathcal{L}_H^\# + \mathcal{L}_R^\#)} \right\rangle [\bullet] = \left\langle \frac{1}{u-(\widetilde{\mathcal{L}}_H + \widetilde{\mathcal{L}}_R)}[\widetilde{\rho}_R^\infty \right\rangle \bullet]. \qquad (46)$$

When the stationary state does not depends on the random rate, $\rho_R^\infty = \rho_S^\infty$, Eq. (46) leads to the conditions

$$\widetilde{\rho}_S^\infty = \rho_S^\infty, \qquad (47a)$$
$$\rho_S^\infty\{\mathcal{L}_H^\# + \mathbb{L}^\#(u)\}[\bullet] = \{\widetilde{\mathcal{L}}_H + \widetilde{\mathbb{L}}(u)\}[\rho_S^\infty\bullet], \qquad (47b)$$

which must to be valid for any value of the Laplace variable u [45]. We notice that a similar structure also arises when formulating the detailed balance condition for non-Markovian classical Fokker-Planck equations [13].

In contrast to the previous conditions [Eq. (44)], Eq. (47) do not depends on the approximations or formalism used to derive the non-Markovian system dynamics. In fact, it only depends on the superoperator $\mathbb{L}(u)$ that defines the density matrix evolution, Eq. (13). In this way we establish a general relation between the non-Markovian QRT and the non-Markovian quantum detailed balance condition. We can affirm that, *whenever the non-Markovian quantum detailed balance conditions Eq. (47) are satisfied, the non-Markovian QRT is fulfilled in the asymptotic regime.* The superoperators $\mathbb{L}^\#(u)$ and $\widetilde{\mathbb{L}}(u)$ follow from $\mathbb{L}(u)$ after replacing all involved superoperators by their dual and time reversed expressions, respectively. In the context of the GBMA, they satisfy Eq. (14) after replacing $\mathcal{L}_H$ and $\mathcal{L}$ by their dual and time reversed expressions.

As we will exemplify in the next section, a typical situation where the non-Markovian QRT is broken, even in the stationary regime, is in systems at thermal equilibrium subject to an external perturbation [9]. In fact, it is possible to prove that conditions Eqs. (47) are not satisfied when a dissipative dynamics that by itself fulfill the detailed balance condition is subject to the action of an external Hamiltonian field that does not commutate with the system Hamiltonian. Equivalently, in the context of the GBMA, the presence of the external perturbation implies that ρ_R^∞ depends on the random rate, which broke the fulfillment of the Markovian conditions Eq. (44).

V. DECAY IN A STRUCTURED THERMAL RESERVOIR

Here we will exemplify our theoretical results by studying a two level system embedded in a complex structured thermal reservoir whose action can be described through the GBMA. The system Hamiltonian is

$$H_S = \frac{\hbar\omega_A}{2}\sigma_z + H_{ext}(t), \qquad (48)$$

where $\hbar\omega_A$ is the difference of energy between the two levels, denoted by $|\pm\rangle$, and σ_z is the z-Pauli matrix. $H_{ext}(t)$ represent an external time dependent field.

The dissipative system dynamics can be defined through the evolution of the states $\rho_R(t)$, which reads

$$\frac{d\rho_R(t)}{dt} = \mathcal{L}_H[\rho_R(t)] + \gamma_R'\mathcal{L}_{th}[\rho_R(t)] + \frac{\gamma_\Phi}{2}\mathcal{L}_\Phi[\rho_R(t)]\}. \qquad (49)$$

with $\mathcal{L}_H[\bullet] = -(i/\hbar)[H_S, \bullet]$. The influence of the structured thermal reservoir is introduced by the Lindblad

superoperator

$$\mathcal{L}_{th}[\bullet] = \frac{1+n_{th}}{2}([\sigma,\bullet\sigma^\dagger]+[\sigma\bullet,\sigma^\dagger]) + \frac{n_{th}}{2}([\sigma^\dagger,\bullet\sigma]+[\sigma^\dagger\bullet,\sigma]), \quad (50)$$

and an arbitrary set $\{\gamma'_R, P_R\}$ of random rates and weights. $\sigma^\dagger$ and σ are the raising and lowering operators acting on the states $|\pm\rangle$. The dimensionless constant n_{th} defines the temperature T of the environment as $\exp[-\hbar\omega_A/kT] = n_{th}/(n_{th}+1)$, where k is the Boltzmann constant. We have also considered the action of an extra dispersive environment which is introduced by the Lindblad superoperator

$$\mathcal{L}_\Phi[\bullet] = ([\sigma_z,\bullet\sigma_z]+[\sigma_z\bullet,\sigma_z])/2, \quad (51)$$

and the single non-random rate γ_Φ.

A. Free decay dynamics

First we analyze the case without the external excitation, i.e., $H_{ext}(t) = 0$.

Density matrix evolution: The evolution of the system density matrix follows from Eq. (13) and (14). By denoting the matrix elements as

$$\rho_S(t) = \begin{pmatrix} \Pi_+(t) & \Phi_+(t) \\ \Phi_-(t) & \Pi_-(t) \end{pmatrix}, \quad (52)$$

in an interaction representation with respect to $\hbar\omega_A\sigma_z/2$, for the populations we get the evolution

$$\frac{d}{dt}\Pi_\pm(t) = \int_0^t d\tau K(t-\tau)\{\mp\Pi_-^{eq}\Pi_+(\tau) \pm \Pi_+^{eq}\Pi_-(\tau)\}, \quad (53)$$

while for the coherences we obtain

$$\frac{d}{dt}\Phi_\pm(t) = -\int_0^t d\tau K_\Phi(t-\tau)\Phi_\pm(\tau). \quad (54)$$

The memory kernel functions are defined by

$$K(u) = \left\langle \frac{\gamma_R}{u+\gamma_R}\right\rangle \left\langle \frac{1}{u+\gamma_R}\right\rangle^{-1}, \quad (55)$$

$$K_\Phi(u) = \left\langle \frac{\gamma_R^\Phi}{u+\gamma_R^\Phi}\right\rangle \left\langle \frac{1}{u+\gamma_R^\Phi}\right\rangle^{-1}. \quad (56)$$

For shortening the notation, we introduced the rates $\gamma_R \equiv \gamma'_R(1+2n_{th})$ and $\gamma_R^\Phi \equiv \gamma_R/2+\gamma_\Phi$. Furthermore, the dimensionless parameters Π_+^{eq} and Π_-^{eq} are defined by $\Pi_+^{eq}/\Pi_-^{eq} = n_{th}/(n_{th}+1)$ and $\Pi_+^{eq}+\Pi_-^{eq} = 1$.

Quantum detailed balance condition: In order to check condition Eq. (47), we note that the evolutions Eq. (53) and (54) can be cast in the superoperator form

$$\mathbb{L}(u)[\bullet] = \frac{1}{1+2n_{th}}K(u)\mathcal{L}_{th}[\bullet] + \frac{K_\phi(u)}{2}\mathcal{L}_\Phi[\bullet], \quad (57)$$

where $K_\phi(u) = K_\Phi(u) - K(u)/2$. The corresponding stationary state reads

$$\rho_S^\infty = \Pi_+^{eq}|+\rangle\langle+| + \Pi_-^{eq}|-\rangle\langle-|, \quad (58)$$

which due to the time reversal invariance of Hamiltonian eigenvectors satisfies $\widetilde{\rho}_S^\infty = \rho_S^\infty$. Then, it is easy to prove that Eq. (47) is satisfied identically. Consistently, notice that the underlying Markovian dynamic Eq. (49) satisfies the conditions Eq. (44).

Quantum regression theorem: As the quantum detailed balance condition is satisfied, the QRT is valid in an asymptotic regime. Consistently the stationary state of Eq. (49) does not depend on γ_R [$\rho_R^\infty = \rho_S^\infty$].

The transient deviation from the QRT can be easily obtained for this example. First, we note that the density matrix evolution defined by Eq. (57) is equivalent to the non-Markovian Bloch equation

$$\frac{dS_X(t)}{dt} = -\int_0^t d\tau K_\Phi(t-\tau)S_X(\tau), \quad (59a)$$

$$\frac{dS_Y(t)}{dt} = -\int_0^t d\tau K_\Phi(t-\tau)S_Y(\tau), \quad (59b)$$

$$\frac{dS_Z(t)}{dt} = -\int_0^t d\tau K(t-\tau)[S_Z(\tau)-\mathcal{S}_Z^\infty], \quad (59c)$$

where $S_j(t) \equiv \mathrm{Tr}_S\{\rho_S(t)\sigma_j\}$ are the expectation values of the Pauli matrixes σ_j, and $\mathcal{S}_Z^\infty \equiv \Pi_+^{eq}-\Pi_-^{eq}$. In order to deal with diagonal matrixes, we analyze the correlations in the base $\mathbf{A} = \{\sigma_x, \sigma_y, (\sigma_z-\mathcal{S}_Z^\infty), \mathrm{I}\}$. Then, the propagator for operator expectation values, $\overline{\mathbf{A}(t)} = \hat{\mathbb{G}}(t)\overline{\mathbf{A}(0)}$, can be written as

$$\hat{\mathbb{G}}(t) = \mathrm{diag}\{P_\Phi(t), P_\Phi(t), P_\Pi(t), 1\}. \quad (60)$$

Here, we defined the functions $P_\Pi(u) = [u+K(u)]^{-1}$ and $P_\Phi(u) = [u+K_\Phi(u)]^{-1}$, which in term of the random rate set can be written in the time domain as

$$P_\Pi(t) = \langle\exp[-\gamma_R t]\rangle, \qquad P_\Phi(t) = e^{-\gamma_\Phi t}P_\Pi(t/2). \quad (61)$$

On the other hand, the extra inhomogeneous term [Eq. (32)] that defines the operator correlations, $\overline{O(t)\mathbf{A}(t+\tau)} = \hat{\mathbb{G}}(\tau)\overline{O(t)\mathbf{A}(t)} + \mathbf{F}(t,\tau)$, can be written as

$$\mathbf{F}(t,\tau) = \hat{\mathbb{G}}_\Pi(t,\tau)\mathbf{F}_\Pi + \hat{\mathbb{G}}_\Phi(t,\tau)\mathbf{F}_\Phi, \quad (62)$$

where we have defined the vectors

$$\mathbf{F}_\Pi = \mathrm{Tr}_S[O(0)\mathbf{A}(0)\{\rho_S^+(0)-\rho_S^\infty\}], \quad (63a)$$

$$\mathbf{F}_\Phi = \mathrm{Tr}_S[O(0)\mathbf{A}(0)\rho_S^-(0)], \quad (63b)$$

with $\rho_S^\pm(0) \equiv [\rho_S(0) \pm \sigma_z\rho_S(0)\sigma_z]/2$ [46]. We note that $\mathbf{F}_\Pi$ measure the departure of the initial populations from the equilibrium values $\Pi_\pm^{eq}$, while $\mathbf{F}_\Phi$ measure the departure of the initial coherences from their null stationary value. Thus, $\mathbf{F}(t,\tau)$ vanishes if the system start in the

equilibrium state ρ_S^∞. On the other hand, the time dependence of $\mathbf{F}(t,\tau)$ is defined by the matrixes

$$\hat{\mathbb{G}}_\Pi(t,\tau) = \mathrm{diag}\{f_0(t,\tau), f_0(t,\tau), f_\Pi(t,\tau), 0\}, \quad (64a)$$
$$\hat{\mathbb{G}}_\Phi(t,\tau) = \mathrm{diag}\{f_\Phi(t,\tau), f_\Phi(t,\tau), f_0(\tau,t), 0\}, \quad (64b)$$

with the definitions

$$f_0(t,\tau) = e^{-\gamma_\Phi \tau} P_\Pi(t+\tau/2) - P_\Pi(t)P_\Phi(\tau), \quad (65a)$$
$$f_\Phi(t,\tau) = P_\Phi(t+\tau) - P_\Phi(t)P_\Phi(\tau), \quad (65b)$$
$$f_\Pi(t,\tau) = P_\Pi(t+\tau) - P_\Pi(t)P_\Pi(\tau). \quad (65c)$$

These functions measure the transient departure from the validity of the QRT. Only when the decay behaviors are exponential, they vanish identically and the QRT is valid at all times. This situation happens when the evolution is Markovian.

B. Transient decay behaviors

In order to illustrate the previous results, we specify the properties of the complex environment, which in the context of the GBMA means to characterize the set $\{\gamma_R, P_R\}$. We choose

$$\gamma_R = \gamma_0 \exp[-bR], \qquad P_R = \frac{(1-e^{-a})}{(1-e^{-aN})} \exp[-aR], \quad (66)$$

where $R \in [0, N-1]$, γ_0 scale the random rates, and the dimensionless constants b and a measure the exponential decay of the random rates and their corresponding weights. The relevant parameters of this set are

$$\gamma \equiv \langle \gamma_R \rangle, \qquad \beta \equiv \frac{\langle \gamma_R^2 \rangle - \langle \gamma_R \rangle^2}{\langle \gamma_R \rangle}, \qquad \alpha \equiv \frac{a}{b}. \quad (67)$$

Here, γ is the average rate and β measures the dispersion of the random rate set. On the other hand, in the limit $N \to \infty$ the set Eq. (66) may leads to system dynamics characterized by a power law behavior whose exponent is given by α [36].

In Fig. 1 we plot the transient decay behavior of the correlation

$$C_{XY}(t,\tau) \equiv \overline{\sigma_x(t)\sigma_y(t+\tau)}, \quad (68)$$

which from Eq. (60) and (62) can be written as

$$C_{XY}(t,\tau) = i\{P_\Phi(\tau)S_Z(t) + f_0(t,\tau)[S_Z(0) - \mathcal{S}_Z^\infty]\}, \quad (69)$$

with $S_Z(t) = \mathcal{S}_Z^\infty + P_\Pi(t)[S_Z(0) - \mathcal{S}_Z^\infty]$. We have chosen a cero temperature reservoir, $n_{th} = 0$, characterized by the random rate set Eq. (66). As initial condition we take the pure state $|+\rangle$. Thus, $\mathcal{S}_Z^\infty = -1$ and $S_Z(0) = 1$. Notice that the initial value of each plot describe the decay of the initial condition from the upper to the lower state. In fact $C_{XY}(t,0) = iS_Z(t)$.

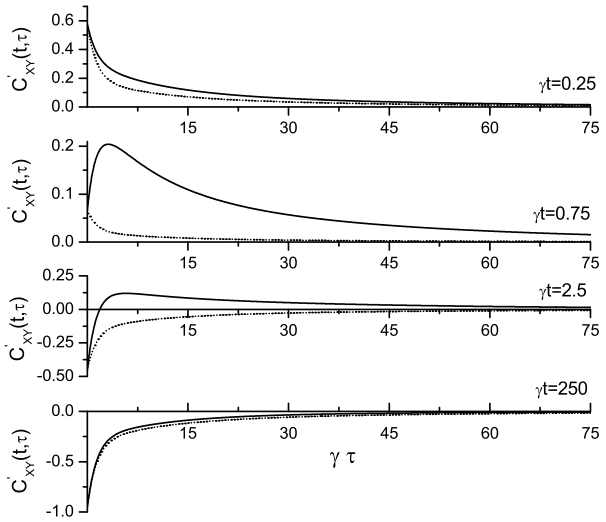

FIG. 1: Transient decay behavior of $C'_{XY}(t,\tau) \equiv C_{XY}(t,\tau)/i$. The parameters of the complex environment are $b = 2.15$, $a = \alpha b$, $\alpha = 1/2$, $N = 5$, $n_{th} = 0$, and $\gamma_\Phi/\gamma = 0.02$. The dispersion rate results $\beta/\gamma = 0.4$. The dotted lines correspond to the QRT. From top to bottom, we set $\gamma t = 0.25$, 0.75, 2.5, and 250.

As can be seen from the graphics, the predictions of the QRT are asymptotically valid in the stationary regime, where the function $f_0(t,\tau)$ vanish identically. In fact, the correlation behavior predicted by the QRT follows from Eq. (69) after replacing $f_0(t,\tau) \to 0$.

The transient deviations from the QRT are proportional to the departure of the system decay behavior from an exponential one. This departure arises from the competence between the exponential decay introduced by the rate γ_Φ and the non-Markovian effects induced by the random rate dispersion. From Eq. (69) it is evident that the dispersive rate γ_Φ introduces a global exponential decay. Thus, in general, by increasing this rate, the transient deviation from the QRT are diminished. On the other hand, an increasing of β implies a strong deviation from an exponential decay.

In order to enlighten the dependence in the dispersion of the random rate set, in Fig. 2 we plot $P_\Phi(t)$, Eq. (61), for different values of the dispersion rate β. This function determine both the coherence decay [46] and the deviations $f_0(t,\tau)$ and $f_\Phi(t,\tau)$, Eq. (65).

The short time behavior can be approximated by the exponential decay $P_\Phi(t) \simeq \exp[-\{\gamma_\Phi + \langle \gamma_R \rangle/2\}t]$, while the asymptotic one by $P_\Phi(t) \simeq \exp[-\{\gamma_\Phi + \langle 1/\gamma_R \rangle/2\}t]$. These behaviors can be straightforwardly obtained from Eq. (61). In the intermediate regime the decay is approximately a power law with exponent α. By diminishing β, the non-exponential decay behaviors occurs at small values of $P_\Phi(t)$. In fact, for $\beta/\gamma \ll 1$, the whole decay may be well approximated by $P_\Phi(t) \simeq \exp[-\{\gamma_\Phi + \langle \gamma_R \rangle/2\}t]$

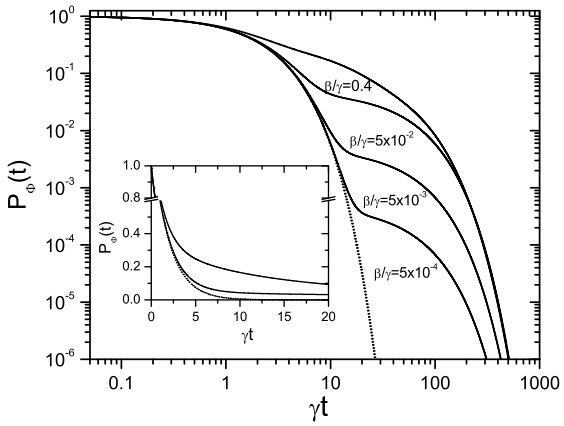

FIG. 2: Decay behavior of $P_\Phi(t)$. From top to bottom, the parameters of the complex environment are $b = 2.15$, 6.05, 10.6 and 15.2. In all cases we take $a = \alpha b$, $\alpha = 1/2$, $N = 5$, $n_{th} = 0$, and $\gamma_\Phi/\gamma = 0.02$. The dotted line corresponds to the Markovian decay $\exp[-t(\gamma_\Phi+\gamma/2)]$. The inset shows the same curves in a linear plot. In this scale, the decay for $b = 10.6$ and 15.2 are indistinguishable from the exponential one.

(see inset). Thus, in this situation, the QRT may be assumed valid at all times for correlations involving the deviations $f_0(t,\tau) \approx 0$ and $f_\Phi(t,\tau) \approx 0$. On the other hand, as the population decay $P_\Pi(t)$ [46] does not involves the dispersive rate γ_Φ, in general we can not disregard the transient effects introduced by $f_\Pi(t,\tau)$, Eq. (65c).

C. Decay under the action of an external field

For dealing with a manageable dynamics, we consider the external Hamiltonian $H_{ext}(t) = (\hbar\Omega/2)(\sigma^\dagger e^{-i\omega_A t} + \sigma e^{+i\omega_A t})$. Then, the system density matrix dynamics can be associated with a spin subject to a resonant external magnetic field [3] or with a two level optical transition driven by a resonant laser field [4].

In an interaction representation with respect to $\hbar\omega_A\sigma_z/2$, the effective system Hamiltonian reads $H_S^{eff} = \hbar\Omega\sigma_x/2$. Thus, the evolution of the states $\rho_R(t)$ is given by Eq. (49) with $H_S \to H_S^{eff}$ (see Appendix A). From Eq. (26a) and (27), the expectation values of the Pauli matrixes evolve as

$$\frac{dS_X(t)}{dt} = -\int_0^t d\tau \Gamma_X(t-\tau) S_X(\tau), \tag{70a}$$

$$\frac{dS_Y(t)}{dt} = -\Omega S_Z(t) - \int_0^t d\tau \{\Gamma_Y(t-\tau) S_Y(\tau) + \Upsilon(t-\tau)[S_Z(\tau) - \mathcal{S}_Z^\infty]\}, \tag{70b}$$

$$\frac{dS_Z(t)}{dt} = \Omega S_Y(t) + \int_0^t d\tau \{\Upsilon(t-\tau) S_Y(\tau) - \Gamma_Z(t-\tau)[S_Z(\tau) - \mathcal{S}_Z^\infty]\}. \tag{70c}$$

In Appendix A we give the exact expressions for the kernels $\Gamma_J(t)$, $j = x, y, z$, and $\Upsilon(t)$, as well as the expression for the non local superoperator $\mathbb{L}(u)$, Eq. (14). We remark that independently of the set of random rates $\{\gamma_R\}$ and weights $\{P_R\}$, the kernels that define the evolution Eq. (70) depend explicitly on the intensity parameter Ω.

The stationary state corresponding to the evolution of each state $\rho_R(t)$, Eq. (49), reads

$$\rho_R^\infty = \frac{1}{2}\left\{ \mathrm{I} + \frac{\Omega\gamma_R\sigma_y - \gamma_R\gamma_R^\Phi\sigma_z}{(1+2n_{th})[\gamma_R\gamma_R^\Phi + \Omega^2]} \right\}, \tag{71}$$

which explicitly depends on γ_R if $\Omega \neq 0$. Then, when

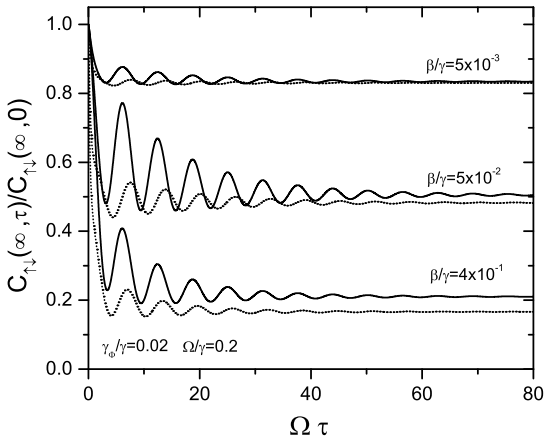

FIG. 3: Stationary decay behavior of $C_{\uparrow\downarrow}(t,\tau)$. From top to bottom, the parameters of the complex environment are $b =$ 10.6, 6.05, and 2.15. In all cases we take $a = \alpha b$, $\alpha = 1/2$, $N = 5$, $n_{th} = 0$, and $\gamma_\Phi/\gamma = 0.02$. The intensity is $\Omega/\gamma = 0.2$. The dotted lines correspond to the QRT.

the system is subject to the action of the external field the QRT in not fulfilled, even in the asymptotic regime. Consistently, as was demonstrated in Ref. [11], the underlying Markovian evolution Eq. (49) does not satisfy the detailed balance condition Eq. (44), neither the superoperator $\mathbb{L}(u)$ [Eq. (A10)] satisfy Eq. (47). As the QRT is not fulfilled, the operators correlations must to be calculated from the microscopic Hamiltonian dynamics, which in our case implies the averaging procedure corresponding to the GBMA.

In the next figures we characterize the correlation

$$C_{\uparrow\downarrow}(t,\tau) \equiv \{C_{XX}(t,\tau) + C_{YY}(t,\tau)\}/4 - i\{C_{XY}(t,\tau) - C_{YX}(t,\tau)\}/4, \tag{72}$$

where $C_{jk}(t,\tau) \equiv \overline{\sigma_j(t)\sigma_k(t+\tau)}$ are the correlations of the Pauli matrixes. Then it follows $C_{\uparrow\downarrow}(t,\tau) = \overline{\sigma^\dagger(t)\sigma(t+\tau)}$. Each contribution $C_{jk}(t,\tau)$ can be determine from Eq. (33), which involves an average of the corresponding Markovian solutions over the random rate

set, Eq. (66). On the other hand, the QRT predictions follows from Eq. (30) with $\mathbf{F}(t,\tau) \to 0$.

In Fig. 3 we plot the stationary decay $C_{\uparrow\downarrow}(\infty,\tau)/C_{\uparrow\downarrow}(\infty,0)$, where $C_{\uparrow\downarrow}(\infty,0) = [1+S_Z(\infty)]/2$, for different values of the rate β. We note that both the decay behaviors and stationary values differ from the QRT predictions. As the evolution of $S_X(t)$ does not depend on Ω [see Eq. (70a)], in the asymptotic regime $C_{XX}(t,\tau)$ satisfies the QRT. Furthermore, as $\lim_{t\to\infty} S_X(t) = 0$, from Eq. (36) and (39) we deduce that the disagreement in the asymptotic values with respect to the QRT only arises due to the contribution $C_{YY}(t,\tau)$, while $C_{XY}(t,\tau)$ and $C_{YX}(t,\tau)$ only contribute to the difference in the decay behaviors.

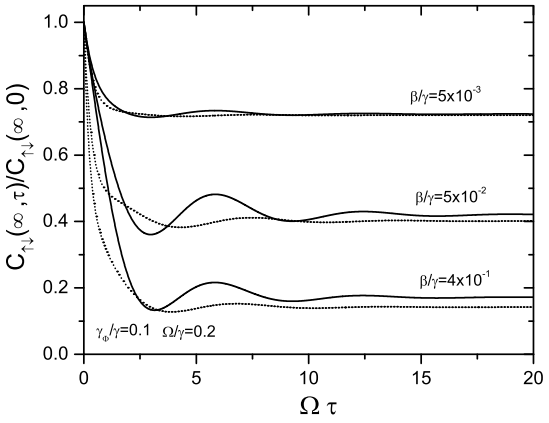

FIG. 4: Stationary decay behavior of $C_{\uparrow\downarrow}(t,\tau)$ after increasing the dispersive rate, $\gamma_\Phi/\gamma = 0.1$. The remaining parameters are the same than in Fig. 3. The dotted lines correspond to the decay predicted by the QRT.

The asymptotic value predicted from Eq. (36) is

$$C_{\uparrow\downarrow}(\infty,\infty) = \frac{1}{(1+2n_{th})^2}\left\langle \left(\frac{\Omega\gamma_R/2}{\gamma_R\gamma_R^\Phi + \Omega^2}\right)^2 \right\rangle, \quad (73)$$

while from Eq. (39), for the QRT we get

$$C_{\uparrow\downarrow}(\infty,\infty) \overset{QRT}{=} \frac{1}{(1+2n_{th})^2}\left\langle \frac{\Omega\gamma_R/2}{\gamma_R\gamma_R^\Phi + \Omega^2} \right\rangle^2, \quad (74)$$

where we have used the stationary state Eq. (71). As can be seen in the graphics, the difference between both predictions grows by increasing the dispersion rate β.

In Fig. 4 , we plot the same correlation after increasing the dispersive rate γ_Φ and maintaining fixed all other parameters. We note that the deviations with respect to the QRT are diminished. In fact, for small values of β/γ, the dynamical deviations goes asymptotically to zero.

In Fig. 5 we plot $C_{\uparrow\downarrow}(\infty,\tau)$ for different values of the field intensity Ω. The deviations with respect to the QRT

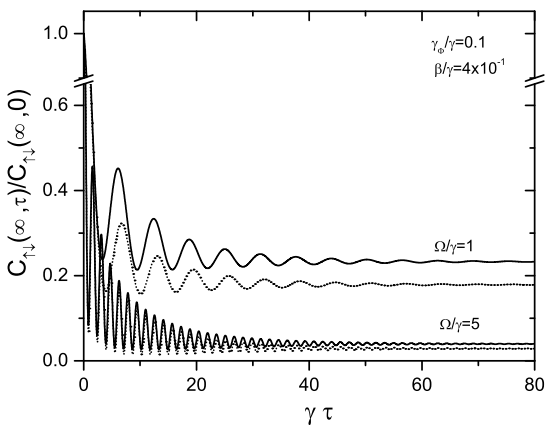

FIG. 5: Stationary decay behavior of $C_{\uparrow\downarrow}(t,\tau)$ for different values of the intensity parameter. From top to bottom, we take $\Omega/\gamma = 1$ and 5. In both cases the parameters of the complex environment are $b = 2.15$, $a = \alpha b$, $\alpha = 1/2$, $N = 5$, $n_{th} = 0$, and $\gamma_\Phi/\gamma = 0.1$. The dotted lines correspond to the decay predicted by the QRT.

are diminished by increasing Ω. Even more, in the limit of high intensity, the dynamical deviations vanish.

The previous parameter dependence analysis relies in a specific correlation and random rate set model. Similar conclusion can be obtained by studying the asymptotic behaviors predicted by Eq. (36) and Eq. (39) for arbitrary correlations and random rate sets. The deviations between both equations are proportional to the random dispersion of the stationary state ρ_R^∞, Eq. (71). Thus, as a measure of the departure from the validity of the QRT in the stationary regime, we introduce the matrix

$$\Xi_{JK} \equiv \left\langle S_J^{(R)}(\infty) S_K^{(R)}(\infty)\right\rangle - \left\langle S_J^{(R)}(\infty)\right\rangle\left\langle S_K^{(R)}(\infty)\right\rangle, \quad (75)$$

with $j,k = x,y,z$, and where $S_j^{(R)}(\infty) \equiv \mathrm{Tr}_S\{\rho_R^\infty \sigma_j\}$ are the random stationary Pauli expectation values. A general characterization of this matrix can be given in a small and high intensity limits.

In the *small intensity limit*, $\Omega \ll \{\gamma_R\}$, we can approximate

$$\Xi_{JK} \approx \frac{\Omega^2}{(1+2n_{th})^2}\{\langle(\tau_R^\Phi)^2\rangle - \langle\tau_R^\Phi\rangle^2\} + O(\Omega^3), \quad (76)$$

where $\tau_R^\Phi \equiv (\gamma_R/2 + \gamma_\Phi)^{-1} = 1/\gamma_R^\Phi$. Consistently, Ξ_{JK} goes to zero in the limit of small intensity Ω. On the other hand, by increasing the dispersive rate γ_Φ, each contribution in Eq. (76) diminish, which in turn means that the predictions of the QRT approach the exact dynamics.

In the *high intensity limit*, $\Omega \gg \{\gamma_R\}$ we get

$$\Xi_{JK} \approx \frac{\Omega^{-2}}{(1+2n_{th})^2}\{\langle\gamma_R^2\rangle - \langle\gamma_R\rangle^2\} + O(\Omega^{-3}). \quad (77)$$

This expression implies that by increasing Ω, the validity of the QRT is asymptotically recuperated. This result is consistent with the fact that at high intensity values [11] the underlying Markovian dynamics for $\rho_R(t)$ satisfies the detailed balance condition Eq. (44). In fact, in this limit the stationary states ρ_R^∞ [Eq. (71)] can be approximated by $\rho_R^\infty \approx \mathrm{I}/2$, which as expected does not depends on the random rate.

VI. SUMMARY AND CONCLUSIONS

In this paper we obtained the conditions under which a QRT can be assumed valid for quantum non-Markovian master equations defined by Lindblad superoperators with memory elements. In order to work on the base of a full Hamiltonian description we deduced our results from a GBMA. This approximation in a natural way leads to these kind of equations. In this context, we demonstrated that operator correlations follow from a weighted average of a set of Markovian solutions, each one characterized by a different dissipative rate.

From our analysis, we deduced that a non-Markovian QRT can only be granted in an stationary regime if the evolution satisfies a non-Markovian quantum detailed balance condition [Eq. (47)], which in turn is related with the time reversal symmetry of the stationary dissipative dynamics. When this is not the case, the QRT is not fulfilled at any time, and in consequence, the only way of calculating operators correlations is from the corresponding microscopic dynamics. We remark that the impossibility of formulating a non-Markovian quantum regression theorem outside a stationary regime can be also demonstrated from general dynamical arguments (see Appendix B).

In general, the departure from the predictions of the QRT not only implies differences in the decay behaviors, but also in the asymptotic values of the operators correlations. The magnitude of these deviations are proportional to the departure of the system dynamics from a semigroup dynamical behavior, i.e., an exponential one.

As an example of our results we worked out the dynamics of a two level system subject to the action of an external coherent field and a complex thermal environment whose action can be described in a GBMA. Without the external field, the QRT is valid in an asymptotic regime. Consistently, the non-Markovian quantum detailed balance condition is also satisfied. The presence of the external field invalidates the QRT, even in the stationary regime. Nevertheless, in the limit of high intensity, or when the effect of a Markovian dispersive contribution is dominant, the QRT is asymptotically reestablished to lowest order in the corresponding expansion parameters.

The present results provides an step forward in the understanding of non-Markovian open quantum systems dynamics. In fact, we have found solid physical criteria for the possibility of using a QRT for calculating operators correlations when the system dynamics is described by a non-local Lindblad evolution.

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APPENDIX A: DENSITY MATRIX EVOLUTION

Here we characterize the evolution of the density matrix elements for the example developed in Section V.

1. Markovian evolution

By denoting the matrix elements as

$$\rho_R(t) = \begin{pmatrix} \Pi_+^{(R)}(t) & \Phi_+^{(R)}(t) \\ \Phi_-^{(R)}(t) & \Pi_-^{(R)}(t) \end{pmatrix}, \tag{A1}$$

in an interaction representation with respect to $\hbar\omega_A\sigma_z/2$, from Eq. (49) we get the evolutions

$$\begin{aligned} \frac{d}{dt}\Pi_\pm^{(R)}(t) &= \gamma_R\{\mp\Pi_-^{eq}\Pi_+^{(R)}(t) \pm \Pi_+^{eq}\Pi_-^{(R)}(t)\} \\ &\quad \pm\frac{i\Omega}{2}[\Phi_+^{(R)}(t) - \Phi_-^{(R)}(t)], \end{aligned} \tag{A2a}$$

$$\frac{d}{dt}\Phi_\pm^{(R)}(t) = \pm\frac{i\Omega}{2}[\Pi_+^{(R)}(t) - \Pi_-^{(R)}(t)] - \gamma_R^\Phi\Phi_\pm^{(R)}(t). \tag{A2b}$$

The operators expectation values are defined by the evolutions

$$\frac{dS_X^{(R)}(t)}{dt} = -\gamma_R^\Phi S_X^{(R)}(t), \tag{A3a}$$

$$\frac{dS_Y^{(R)}(t)}{dt} = -\Omega S_Z^{(R)}(t) - \gamma_R^\Phi S_Y^{(R)}(t), \tag{A3b}$$

$$\frac{dS_Z^{(R)}(t)}{dt} = \Omega S_Y^{(R)}(t) - \gamma_R[S_Z^{(R)}(t) - \mathcal{S}_Z^\infty], \tag{A3c}$$

where $S_j^{(R)}(t) \equiv \mathrm{Tr}_S\{\rho_R(t)\sigma_j\}$, with σ_j the Pauli matrixes.

2. Non-Markovian evolution

Here, for arbitrary set $\{\gamma_R, P_R\}$, we present the exact expressions for the kernels that define the evolution

Eq. (70). From Eq. (27) and Eq. (A3), we get

$$\Gamma_X(u) = K_\Phi(u), \tag{A4a}$$
$$\Gamma_Y(u) = D\{(u+C)[\frac{B}{2}+(u+\gamma_\Phi)]+\Omega^2\}+\gamma_\Phi, \tag{A4b}$$
$$\Gamma_Z(u) = 2D\{(u+B)[\frac{C}{2}+(u+\gamma_\Phi)]+\Omega^2\}, \tag{A4c}$$
$$\Upsilon(u) = D(C-B)\Omega, \tag{A4d}$$

where D denotes the function

$$D(u) = \frac{B(u)/2}{[u+B(u)][u+B(u)/2+\gamma_\Phi]+\Omega^2}. \tag{A5}$$

The extra function B and C are defined by

$$B(u) = \frac{\langle T(u)\gamma_R\rangle}{\langle T(u)\rangle}, \qquad C(u) = \frac{\langle T(u)(\gamma_R)^2\rangle}{\langle T(u)\gamma_R\rangle} \tag{A6}$$

where we have introduced

$$T(u) = \frac{1/2}{(u+\gamma_R)[u+\gamma_R/2+\gamma_\Phi]+\Omega^2}. \tag{A7}$$

The corresponding density matrix evolution can be written as

$$\dot{\Pi}_\pm(u) = \Gamma_Z(u)\{\mp\Pi_-^{eq}\Pi_+(u)\pm\Pi_+^{eq}\Pi_-(u)\}, \pm\frac{i\Omega(u)}{2}[\Phi_+(u)-\Phi_-(u)], \tag{A8a}$$
$$\dot{\Phi}_\pm(u) = \pm\frac{i\Omega(u)}{2}[\Pi_+(u)-\Pi_-(u)]\pm i\frac{\Upsilon(u)/(2u)}{1+2n_{th}} -\Gamma_\Phi^+(u)\Phi_\pm(u)-\Gamma_\Phi^-(u)\Phi_\mp(u), \tag{A8b}$$

where, for shortening the notation, $\dot{g}(u)\equiv ug(u)-g(0)$ denotes the Laplace transform of the time derivative of an arbitrary function $g(t)$. In the inhomogeneous term for the coherences, we have used $\Pi_+(u)+\Pi_-(u)=1/u$. Furthermore, we have defined $\Omega(u)\equiv\Omega+\Upsilon(u)$ and $\Gamma_\Phi^\pm(u)\equiv\frac{1}{2}[\Gamma_X(u)\pm\Gamma_Y(u)]$. The stationary state reads

$$\rho_S^\infty = \frac{1}{2}\left\{\mathrm{I}+\frac{\Omega\Gamma_Z\sigma_y-[\Gamma_Y\Gamma_Z+\Upsilon(\Upsilon+\Omega)]\sigma_z}{(1+2n_{th})[\Gamma_Y\Gamma_Z+(\Upsilon+\Omega)^2]}\right\}, \tag{A9}$$

with the notation $\Gamma_J\equiv\Gamma_J(u)|_{u=0}$. Consistently, after some algebra, it is possible to write this state as an average of the corresponding Markovian stationary states, i.e., $\rho_S^\infty=\langle\rho_R^\infty\rangle$, where ρ_R^∞ is defined by Eq. (71).

The superoperator $\mathbb{L}(u)$ [Eq. (14)] corresponding to the evolution Eq. (A8) can be written as non-diagonal non-local Lindblad superoperator

$$\mathbb{L}(u)[\bullet] = \mathcal{L}_H(u)[\bullet]+\frac{1}{2}\sum_{\alpha\beta}a_{\alpha\beta}(u)([V_\alpha,\bullet V_\beta^\dagger]+[V_\alpha\bullet,V_\beta^\dagger]), \tag{A10}$$

with the operators $\{V_\alpha\}_{\alpha=1,2,3}=\{\sigma,\sigma^\dagger,\sigma_z\}$. The Hamiltonian contribution reads

$$\mathcal{L}_H(u)[\bullet] = -i\frac{\Upsilon(u)}{2}[\sigma_x,\bullet], \tag{A11}$$

and the matrix elements $a_{\alpha\beta}(u)$ are defined by

$$a_{11}(u) = \Pi_-^{eq}\Gamma_Z(u), \tag{A12}$$
$$a_{22}(u) = \Pi_+^{eq}\Gamma_Z(u), \tag{A13}$$
$$a_{33}(u) = \frac{1}{4}\{\Gamma_X(u)+\Gamma_Y(u)-\Gamma_Z(u)\}, \tag{A14}$$
$$a_{12}(u) = a_{21}(u) = -\frac{1}{2}\{\Gamma_X(u)-\Gamma_Y(u)\}, \tag{A15}$$
$$a_{13}(u) = a_{23}(u) = -i\frac{\Upsilon(u)}{4(1+2n_{th})}, \tag{A16}$$
$$a_{31}(u) = a_{32}(u) = i\frac{\Upsilon(u)}{4(1+2n_{th})}. \tag{A17}$$

Without the external excitation, $\Omega=0$, the superoperator $\mathbb{L}(u)$ reduce to Eq. (57). Furthermore, when the coherence decay behavior can be approximated by an exponential one, $P_\Phi(t)=e^{-\gamma_\Phi t}P_\Pi(t/2)\simeq\exp[-(\gamma_\Phi+\langle\gamma_R\rangle/2)t]$, the density matrix evolution can be written in a Schrödinger representation as

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{\hbar}[H_S,\rho_S(t)]+\frac{\gamma_\Phi}{2}\mathcal{L}_\Phi[\rho_S(t)] +\frac{1}{1+2n_{th}}\int_0^t d\tau K(t-\tau)\mathcal{L}_{th}[\rho_S(\tau)], \tag{A18}$$

with $H_S=\hbar\omega_A\sigma_z/2$. This expression relies in the validity of the approximation $K(u\pm iw_A)\simeq K(\infty)=\langle\gamma_R\rangle$, which can be considered always valid if w_A is an optical frequency. Furthermore, if $\gamma_\Phi\ll\langle\gamma_R\rangle$ the dispersive contribution can be drop. In general this last condition is valid when the decay of $P_\Pi(t)$ develops two strong different time scales. For example, consider a random rate that assumes only two different values $\gamma_{\uparrow/\downarrow}$, with probabilities $P_{\uparrow/\downarrow}$. Then $P_\Pi(t)=P_\uparrow e^{-\gamma_\uparrow t}+P_\downarrow e^{-\gamma_\downarrow t}$. Under the conditions $P_\downarrow\ll P_\uparrow$ and $\gamma_\downarrow\ll\gamma_\Phi\ll\gamma_\uparrow$, we can approximate $P_\Pi(t)e^{-\gamma_\Phi t}\approx e^{-\langle\gamma_R\rangle t}+O(P_\downarrow/P_\uparrow)$. Another examples follow from the decay of Fig. 2 for small β/γ. On the other hand, Eq. (A18) can also be assumed valid in presence of the external field if the exact kernels are taken to cero order in the intensity parameter Ω.

APPENDIX B: ON THE IMPOSSIBILITY OF FORMULATING A NON-MARKOVIAN QUANTUM REGRESSION THEOREM AT ALL TIMES

The impossibility of formulating a non-Markovian regression theorem outside a stationary regime can be demonstrated on general dynamical arguments. In fact, it is simple to proof that the validity of the quantum regression theorem at all times is *only* compatible with a Markovian dynamics. This affirmation seems to contradict our main conclusions. Nevertheless, here we demonstrate that this result confirm the correctness of our approach.

First, we write the system density matrix as

$$\rho_S(t) = \mathbb{T}(t)[\rho_S(0)], \tag{B1}$$

where $\mathbb{T}(t)$ is the propagator corresponding to the evolution Eq. (13). Then, it is defined in the Laplace domain by

$$\mathbb{T}(u) = \frac{1}{u - [\mathcal{L}_H + \mathbb{L}(u)]}. \tag{B2}$$

In terms of this object, we can write the operator expectation values as

$$\begin{aligned} \overline{\mathbf{A}(t)} &= \mathrm{Tr}_S\{\mathbf{A}(0)\mathbb{T}(t)[\rho_S(0)]\}, & \text{(B3a)} \\ &= \mathrm{Tr}_S\{\rho_S(0)\mathbb{T}^{\#}(t)[\mathbf{A}(0)]\}, & \text{(B3b)} \end{aligned}$$

where the second line defines the dual propagator $\mathbb{T}^{\#}(t)$. By assuming valid the quantum regression theorem, the operator correlations can be written as [4, 5, 6, 7]

$$\overline{O(t)\mathbf{A}(t+\tau)} = \mathrm{Tr}_S\{\rho_S(t)O(0)\mathbb{T}^{\#}(\tau)[\mathbf{A}(0)]\}. \tag{B4}$$

This expression must to be valid for arbitrary operators O and $\mathbf{A}$. In particular, by taking $O = I_S$, where I_S is the system identity operator, it follows

$$\begin{aligned} \overline{\mathbf{A}(t+\tau)} &= \mathrm{Tr}_S\{\rho_S(t)\mathbb{T}^{\#}(\tau)[\mathbf{A}(0)]\}, & \text{(B5a)} \\ &= \mathrm{Tr}_S\{\mathbf{A}(0)\mathbb{T}(\tau)[\rho_S(t)]\}, & \text{(B5b)} \\ &= \mathrm{Tr}_S\{\mathbf{A}(0)\mathbb{T}(\tau)\mathbb{T}(t)[\rho_S(0)]\}. & \text{(B5c)} \end{aligned}$$

On the other hand, from Eq. (B3a), we can write

$$\overline{\mathbf{A}(t+\tau)} = \mathrm{Tr}_S\{\mathbf{A}(0)\mathbb{T}(t+\tau)[\rho_S(0)]\}. \tag{B6}$$

As $\mathbf{A}(0)$ is an arbitrary operator, by comparing this expression and Eq. (B5c), it follows

$$\mathbb{T}(t+\tau)\rho_S(0) = \mathbb{T}(\tau)\mathbb{T}(t)\rho_S(0). \tag{B7}$$

For arbitrary time $t < \infty$, and $\rho_S(0) \neq \rho_S^{\infty}$, where ρ_S^{∞} is the stationary state corresponding to the dynamics Eq. (B1), this equality can only be satisfied if the propagator $\mathbb{T}(t)$ corresponds to a semigroup structure, i.e., a *Markovian evolution*. Therefore, a regression theorem can be satisfied at all times only when the dynamics does not has any memory contribution. We notice that this result is in perfect agreement with our main conclusions. In fact, we have found that a non-Markovian quantum regression theorem may be valid (or not) *only in a stationary regime*. In this limit, the previous calculations steps *does not impose* any constraint on the propagator $\mathbb{T}(t)$. This affirmation follows *trivially* by taking $\rho_S(0) = \rho_S^{\infty}$ in Eq. (B7), or equivalently by introducing the limit $t \to \infty$,

$$\lim_{t\to\infty} \mathbb{T}(t+\tau)\rho_S(0) = \lim_{t\to\infty} \mathbb{T}(\tau)\mathbb{T}(t)\rho_S(0), \tag{B8}$$

which, independently of the properties of $\mathbb{T}(t)$, deliver $\rho_S^{\infty} = \rho_S^{\infty}$. This last equality follows immediately from $\mathbb{T}(\tau)[\rho_S^{\infty}] = \rho_S^{\infty}$, expression valid for any time τ. Alternatively, one can take the limit $t \to \infty$ in Eq. (B5c)

$$\begin{aligned} \lim_{t\to\infty} \overline{\mathbf{A}(t+\tau)} &= \mathrm{Tr}_S\{\mathbf{A}(0)\mathbb{T}(\tau) \lim_{t\to\infty} \mathbb{T}(t)[\rho_S(0)]\}, & \text{(B9a)} \\ &= \mathrm{Tr}_S\{\mathbf{A}(0)\mathbb{T}(\tau)[\rho_S^{\infty}]\}, & \text{(B9b)} \\ &= \mathrm{Tr}_S\{\mathbf{A}(0)\rho_S^{\infty}\}. & \text{(B9c)} \end{aligned}$$

On the other hand, in the same limit, from Eq. (B6), as expected, we get the same result

$$\lim_{t\to\infty} \overline{\mathbf{A}(t+\tau)} = \mathrm{Tr}_S\{\mathbf{A}(0)\rho_S^{\infty}\}. \tag{B10}$$

Therefore, the calculations steps that lead to the constraint Eq. (B7) only contradict the possibility of establishing a non-Markovian quantum regression theorem outside the stationary regime. These arguments provide an alternative demonstration of the consistency and correctness of our results.

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[40] In Ref. [35] a similar conclusion was obtained from an abstract tripartite system-environment interaction. Here we proved this result from an usual bipartite interaction.
[41] If the set $\{g_{nm}\}$ does not depends on the parameter ε, by assuming valid the detailed balance condition Eq. (41), one can find a continuous parametrized infinite family of stationary solutions. In fact, in such a case, the detailed balance condition can be written as $g_{nm}\sum_{k=0}^{\infty} p_m^{(k)}(\varepsilon-\varepsilon_0)^k/k! = g_{mn}\sum_{k=0}^{\infty} p_n^{(k)}(\varepsilon-\varepsilon_0)^k/k!$, where $p_n^{(k)} \equiv (\partial^k/\partial\varepsilon^k)p_n(\infty,\varepsilon)|_{\varepsilon=\varepsilon_0}$. As these relations must to be valid for any ε, it follows that for any k and ε_0 the set of derivatives $\{p_n^{(k)}\}$ allows to construct a new and different stationary solution.
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[44] By writing the Lindblad superoperator Eq. (12) in the form $\mathcal{L}[\bullet] = \sum_{\alpha\beta} a_{\alpha\beta}(V_\alpha \bullet V_\beta^\dagger - \frac{1}{2}\{V_\beta^\dagger V_\alpha, \bullet\}_+)$, the dual superoperator $\mathcal{L}^{\#}$ can be immediately expressed as $\mathcal{L}^{\#}[\bullet] = \sum_{\alpha\beta} a_{\alpha\beta}(V_\beta^\dagger \bullet V_\alpha - \frac{1}{2}\{V_\beta^\dagger V_\alpha, \bullet\}_+)$, where $\{\cdot,\cdot\}_+$ denotes an anticonmutator operation. For the Hamiltonian contribution if follows $\mathcal{L}_S^{\#} = -\mathcal{L}_S$. On the other hand, $\widetilde{\mathcal{L}}_H = -\mathcal{L}_H$, and when $\widetilde{V}_\alpha = V_\alpha^\dagger$, we get $\widetilde{\mathcal{L}} = \mathcal{L}$.
[45] As in Eq. (44), the condition Eq. (47b) can be split in two equations, corresponding respectively to the Hamiltonian and dissipative contributions. Nevertheless, in the non-Markovian case we must also to consider possible non-local Hamiltonian contributions introduced by $\mathbb{L}(u)$ [see for example Eq. (A10)].
[46] The solutions of Eq. (53) and (54) can be written as $\Pi_\pm(t) = P_\Pi(t)\Pi_\pm(0) + [1-P_\Pi(t)]\Pi_\pm^{eq}$, while for the coherences it reads $\Phi_\pm(t) = P_\Phi(t)\Phi_\pm(0)$. From these expressions, it is possible to write the system density matrix as $\rho_S(t) = P_\Pi(t)\rho_S^+(0) + P_\Phi(t)\rho_S^-(0) + [1-P_\Pi(t)]\rho_S^\infty$. This final expression is used when deriving Eq. (62).